A Discussion of the Connectedness Locus of Complex Cubic Polynomials.

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Abstract

With computers, we are able to construct complicated fractal images that describe the dynamics of complex polynomials. Much is known about the Mandelbrot set and how it classifies the dynamics of complex quadratic polynomials, but less is known about the dynamics of complex cubic polynomials. This paper discusses the Connectedness Locus of a family of complex cubic polynomials, and how its geometric structure describes the dynamics of this family.
The study of abstract dynamical systems made its first significant appearance over a century ago with the work of the French mathematical physicist Henri Poincare in the late 1800’s, but it wasn’t until years later, in the 1970’s, that Benoit Mandelbrot, with the aid of computers, was able to show how chaotic dynamical systems are the source of some of the most beautiful, natural images in the world. Since Mandelbrot’s discovery, much analysis has been done on his famed Mandelbrot Set (see figure 1). The Mandelbrot Set is the connectedness locus for complex quadratic polynomials. Much is known about it and its corresponding Julia Sets.

More recently, with the works of such mathematicians as Linda Keen, Robert Devaney, Bodil Branner, and John Hubbard, the study of complex cubic polynomials is becoming somewhat of a large interest. However, much less is known about the dynamics of complex cubic polynomials than the dynamics of complex quadratic polynomials.

This paper serves as an introduction to the study of the dynamics of complex cubic polynomials. More specifically, it deals with the connectedness locus for a specific one-parameter family of cubics, and it gives a brief discussion of the dynamics of this family. Reference will occasionally be made to the dynamics of complex quadratic polynomials for a better understanding of the methods being used.

Discrete dynamical systems are created through iteration. In other words, in dynamical systems, each step is a result of the step directly before it. A
good example to consider is the exponential function $e^x$. Beginning with a value $x$, to iterate it under $f(x) = e^x$ means to get a series of values as such,

$$x, \quad e^x, \quad e^{e^x}, \quad e^{e^{e^x}}, \ldots$$

Before moving forward, there are a couple of definitions that should be known and understood.

**Definition 1.** \{x, f(x), f(f(x)), f(f(f(x))), ...\} is called the orbit of $x$ under $f(x)$.

**Definition 2.** The initial value, $x$, is called the seed of the function.

The main question that should be asked when studying the dynamics of a system is: Given a function $f$ and an initial seed $x_0$, what happens to its orbit?

We will consider all quadratic polynomials to start. Since there are so many forms that a quadratic formula may take on, it is convenient to reduce them all to a form that is easier to understand. It turns out that the general form for quadratic polynomials,

$$a_2 z^2 + a_1 z + a_0$$

can be conjugated to $P_a(z) = z^2 + c$ using an affine change of the $z$-variable. In other words there exists a map $h(z) = az + b$ such that $h^{-1} \circ P \circ h = P_c$.[1]
The mapping is represented below:

\[
\begin{align*}
\mathbb{C} & \xrightarrow{P(z)=a_2z^2+a_1z+a_0} \mathbb{C} \\
\downarrow (az+b) & \\
\mathbb{C} & \xrightarrow{P_c(z)=z^2+c} \mathbb{C}
\end{align*}
\]

In order to define these variables, one must solve the equation:

\[(az + b)^2 + c = a(a_2x^2 + a_1x + a_0) + b\]

This is true when,

\[
\begin{align*}
a &= a_2 \\
b &= \frac{a_1}{2} \\
c &= \frac{a_1}{2} + a_2a_0 - \left(\frac{a_1}{2}\right)^2
\end{align*}
\]

So, all quadratic polynomials are equivalent to \(z^2 + c\) for some \(c\).

Similarly, all complex cubic polynomials are conjugate to the function

\[P_{a,b}(z) = z^3 - 3a^2z + b\]

for some parameters \(a\) and \(b\). The conjugation map for cubics is represented below:

\[
\begin{align*}
\mathbb{C} & \xrightarrow{P(z)=a_3z^3+a_2z^2+a_1z+a_0} \mathbb{C} \\
\downarrow (cz+d) & \\
\mathbb{C} & \xrightarrow{P_{a,b}(z)=z^3-3a^2z+b} \mathbb{C}
\end{align*}
\]

In order to define these variables, one must solve the equation:
\[(cz + d)^3 - 3z^2(cz + d) + b = c(a_3z^3 + a_2z^2 + a_1z + a_0) + d\]
\[(c^3)z^3 + (3c^2d)z^2 + (3cd^2 - 3a^2c)z + (d^3 - 3a^2d + b) =\]
\[(ca_3)z^3 + (ca_2)z^2 + (ca_1)z + (ca_0 + d)\]

This is true when,
\[c^3 = ca_3\]
\[3c^2d = ca_2\]
\[3cd^2 - 3a^2c = ca_1\]
\[d^3 - 3a^2d + b = ca_0 + d\]

Notice:
\[c^3 = ca_3 \Rightarrow c^2 = a_3 \Rightarrow c = \pm \sqrt[3]{a_3}\]
\[3c^2d = ca_2 \Rightarrow 3cd = a_2 \Rightarrow d = \frac{a_2}{3c}\]
\[3cd^2 - 3a^2c = ca_1 \Rightarrow 3d^2 - 3a^2 = a_1 \Rightarrow -3a^2 = a_1 - 3d^2\]
\[d^3 - 3a^2d + b = ca_0 + d \Rightarrow b = ca_0 + d - d^3 + 3a^2d\]

So all cubic polynomials are conjugate to \(z^3 - 3a^2z + b\) for some complex values of \(a\) and \(b\).

It has been established, by the work of Gaston Julia([4]) and Pierre Fatou([3]) in the early part of the 20\(^{th}\) century, that for any discrete dynamical system described by a polynomial, \(F : \mathbb{C} \rightarrow \mathbb{C}\), the plane can be decomposed into two disjoint regions.
• Points on the complex z plane (dynamical or variable space) which under iterations go to an attractor (attracting fixed point, periodic orbit or infinity) form the Fatou set.[5]

• The Julia set is its complement, consisting of all other points where the function is chaotic and difficult to describe.

For example, consider $f(z) = z^2$.

Every element outside of the unit circle tends to infinity under successive iterations. Every element within the unit circle tends to zero under successive iterations. The elements of the unit circle behave entirely chaotically. In this specific case, the Fatou set would consist of the entire plane minus the unit circle and the Julia set would be the unit circle.

It is important to note that the variable $z$ and the parameter $c$ both fill out a plane. For a fixed $c$ the $z$-plane is known as the dynamical plane for $P_c$, while the $c$ plane is known as the parameter plane. It is important to realize that every point in the parameter plane gives rise to its own unique picture in the dynamical plane.

Fatou and Julia also show that for a general polynomial,

• If the orbits of all critical points are bounded then the Julia set is connected.

• If the orbits of all critical points are unbounded then the Julia set is a totally disconnected Cantor set.
Since $P_c(z) = z^2 + c$ has only one critical point, this leads to the basic dichotomy:

Its Julia set is either connected or totally disconnected.

In order to study $P_c(z) = z^2 + c$, we look at the orbit of 0, 

$$\{c, c^2 + c, (c^2 + c)^2 + c, \ldots\}, \text{ for all } c \in \mathbb{C}$$

The Mandelbrot Set ($M$) is the set of all parameter values where successive iterations under $P_c(z)$ do not tend toward infinity,

$$M = \{c \in \mathbb{C} | c, c^2 + c, (c^2 + c)^2 + c, \ldots \nrightarrow \infty\}$$

As stated earlier, $M$ is the connectedness locus for all quadratic polynomials and lives in the parameter plane. If $c \in M$ then the Julia set corresponding to that particular parameter will be connected. If it is the case that $c \notin M$, then the Julia set is totally disconnected.

We will now focus more on complex cubic polynomials. We have shown that any cubic polynomial is conjugate to one of the form,

$$P_{a,b}(z) = z^3 - 3a^2z + b,$$ for some $a$ and $b$

Since there are two complex parameters, the connectedness locus of this equation for cubics will live in $\mathbb{C} \times \mathbb{C}$, which is a four dimensional space. Furthermore, because there are two critical points, the basic dichotomy fails. In its place is a basic trichotomy.
• If the orbit of both critical points tend toward 0, then the corresponding Julia set is connected.

• If the orbit of both critical points tend toward infinity, then the corresponding Julia set is totally disconnected.

• But, if the orbit of one critical point tends to 0 while the other’s orbit tends to infinity, then the corresponding Julia set will have an infinite number of components, which are not, in general, a single element.

For these reasons, we will limit ourselves to a two-dimensional cross-section, the one parameter family of cubics obtained by setting \( b = 0 \). That is we are going to deal with the family of functions:

\[
P_a(z) = z^3 - 3a^2z
\]

Dealing with this family of cubics is quite an advantage. There are a number of very interesting facts that are to ensue as a result of dealing with this family.

1. \( P_a(-z) = -P_a(z) \)

\[
P_a(-z) = (-z^3) - 3a^2(-z)
\]

\[
= -z^3 + 3a^2z
\]

\[
= -(z^3 - 3a^2z)
\]

\[
= -P_a(z)
\]
The critical points of $P_a(z)$ are $a$ and $-a$

Therefore, $\text{orbit}(-a) = -\text{orbit}(a)$

The orbit of one critical point will always act in the same way as the other. Therefore the dichotomy holds.

2. $P_{-a}(z) = P_a(z)$

\[
P_{-a}(z) = z^3 - 3(-a)^2z
= z^3 - 3a^2z
= P_a(z)
\]

This implies that the connectedness locus, $C$, is symmetric about the origin.

3. $P_a(z) = \overline{P_a(\overline{z})}$

\[
\overline{P_a(\overline{z})} = \overline{z^3 - 3a^2\overline{z}}
= \overline{z^3} - 3a^2\overline{z}
= z^3 - 3\overline{a}^2\overline{z}
= \overline{P_a(z)}
\]

This implies that $C$ is symmetric about the x-axis.

4. Furthermore, 2 and 3 imply that $C$ is symmetric about the y-axis.
Let’s begin to describe the structure of $C$ and how it corresponds to the dynamics of the polynomial. A picture of $C$ is given in figure 2. To begin, we’ll want to find all fixed points of $P_a(z)$, that is to say that we want to solve the equation $P_a(z) = z$

\[ P_a(z) = z \]
\[ z^3 - 3a^2 = z \]
\[ z^3 - 3a^2 - z = 0 \]
\[ z(z^2 - 3a^2 - 1) = 0 \]

So, we have $z = 0$ as a fixed point. Now, to find the other two:

\[ z^2 - 3a^2 - 1 = 0 \]
\[ z^2 = 3a^2 + 1 \]
\[ z = \pm \sqrt{3a^2 + 1} \]

So, the fixed points are 0, $\sqrt{3a^2 + 1}$, $-\sqrt{3a^2 + 1}$

When are they attracting fixed points? In other words, is there a neighborhood around the fixed points where any element of that neighborhood tends toward the particular fixed point? The following theorem and proof come directly from Devaney’s text on page 276, cited as [2].
Theorem 1. Let $P(z)$ be an analytic function satisfying $P(0) = 0$ and $P'(0) = \lambda$ with $0 < |\lambda| < 1$. Then there is a neighborhood $U$ of 0 and an analytic map $H : U \to \mathbb{C}$ such that

$$P \circ H(z) = H \circ L(z)$$

where $L(z) = \lambda z$.

Remark. The functional equation $P \circ H = H \circ L$ is called the Shroder functional equation.

Proof. We may write $P$ as a power series in the form

$$P(z) = \lambda z + \sum_{l=2}^{\infty} a_l z^l$$

where this series converges in some neighborhood of 0. We must therefore find an analytic function $H(z)$ that solves the functional equation $P \circ H = H \circ L$. Such an analytic function must be represented as a power series and so we may write

$$H(z) = \sum_{l=0}^{\infty} c_l z^l.$$

We will first determine what the coefficients $c_l$ of this power series must be. This is a formal solution of the functional equation. Then we must show that the series defined actually converges in some neighborhood of 0.

Determining the formal solution is the easy part. Since both $P$ and $L$ fix 0, so too must $H$. This means that $c_0 = 0$. Next, comparing the first order
terms of $P(H(z))$ and $H(\lambda z)$, we see that $c_1$ may be chosen arbitrarily (but non-zero). We then set $c_1$, so that

$$H(z) = z + \sum_{l=2}^{\infty} c_l z^l.$$  

We now proceed to determine the remaining $c_l$ recursively. The functional equation may be written

$$\lambda H(z) + \sum_{l=2}^{\infty} a_l(H(z))^l = \lambda z + \sum_{l=2}^{\infty} c_l \lambda^l z^l.$$

Therefore we have

$$\sum_{l=2}^{\infty} (\lambda^l - \lambda)c_l z^l = \sum_{l=2}^{\infty} a_l(H(z))^l.$$

Suppose that $c_2, ..., c_{k-1}$ are known. Then the $k^{th}$ order terms on the right hand side of this equation involve only $c_2, ..., c_{k-1}$ since that summation begins with terms of order two. In particular, the coefficient of $z^k$ on the right is a polynomial in $a_2, ..., a_k$ as well as $c_2, ..., c_{k-1}$ which has positive integer coefficients. We denote this term by $K_k(a_2, ..., a_k, c_2, ..., c_{k-1})$ and emphasize that, as long as $c_2, ..., c_{k-1}$ have been determined, $k_k$ is known.

Then, comparing coefficients, we have

$$C_k = \frac{K_k(a_2, ..., a_k, c_2, ..., c_{k-1})}{\lambda^k - \lambda}$$

Thus, $c_k$ is determined by the previous coefficients, provided $\lambda^k \neq \lambda$, and we have our formal solution.
Before proving that this solution converges, we observe that the expression for \( c_k \) shows why this procedure breaks down when \( \lambda = 0 \) or \( \lambda = (k-1)^{st} \) root of unity. In either case, the expression for \( c_k \) is undefined.

To prove that the power series

\[
H(z) = \sum_{l=2}^{\infty} c_l z^l
\]

converges, several lemmas are needed.

For the purposes of this paper, it is sufficient to know that the power series does converge. The proofs of the lemmas can be found in Devaney’s text [2].

So we have that the fixed points are attracting when \( |P_a'(z_0)| < 1 \) for a fixed point \( z_0 \), and \( P_a'(z) = 3z^2 - 3a^2 \).

\[
|P_a'(0)| < 1
\]

\[
|3(0)^2 - 3a^2| < 1
\]

\[
|-3a^2| < 1
\]

\[
|a^2| < \frac{1}{3}
\]

\[
|a| < \frac{1}{\sqrt{3}}
\]

So the fixed point, 0, is attracting when \( |a| < \frac{1}{\sqrt{3}} \). This is a circle with center 0 and radius \( \frac{1}{\sqrt{3}} \).

Now consider the other two fixed points. Notice that because of the symmetry that exists, \( \sqrt{3a^2 + 1} \) and \( -\sqrt{3a^2 + 1} \) may be dealt with at the
same time.

\[|P_a(\pm \sqrt{3a^2 + 1})| = |3(\pm \sqrt{3a^2 + 1})^2 - 3a^2|\]
\[= |3(3a^2 + 1) - 3a^2|\]
\[= |9a^2 + 3 - 3a^2|\]
\[= |6a^2 + 3|\]
\[|6a^2 + 3| < 1\]
\[|a^2 + \frac{1}{2}| < \frac{1}{6}\]

Let \(w = a^2\). So:

\[|w + \frac{1}{2}| < \frac{1}{6}\]

This is a circle with center at \(-\frac{1}{2}\) and radius \(\frac{1}{6}\). But we want \(a^2\), not \(w\). So we have to take the square root of the circle \(|w + \frac{1}{2}| < \frac{1}{6}\). Doing so will yield two lobes “centered” at \(\frac{1}{\sqrt{2}}i\) and \(-\frac{1}{\sqrt{2}}i\), both tangent to the circle \(|a| < \frac{1}{\sqrt{3}}\) at the points \(\frac{1}{\sqrt{3}}i\) and \(-\frac{1}{\sqrt{3}}i\) respectively, and both with maximum “heights” at \(\frac{\sqrt{6}}{3}i\) and \(-\frac{\sqrt{6}}{3}i\) respectively (see figure 7).

So, we’ve algebraically and graphically found all three attracting fixed point lobes. Now we will move on to find general formulas for two cycle points. That is, we want to find all points that satisfy the equation

\[P_a(P_a(z)) = z\]
or

\[(z^3 - 3a^2z)^3 - 3a^2(z^2 - 3a^2z) = z\]

Since every fixed point will satisfy this equation, we already have three solutions to this equation: 0, \(\sqrt{3a^2 + 1}\), and \(-\sqrt{3a^2 + 1}\). So, since it is a ninth degree polynomial, there are six more solutions to find.

\[(z^3 - 3a^2z)^3 - 3a^2(z^2 - 3a^2z) = z\]
\[(z^3 - 3a^2z)^3 - 3a^2(z^2 - 3a^2z) - z = 0\]
\[((z^2 - 3a^2)z)^3 - 3a^2(z - 3a^2)z - z = 0\]
\[z[(z^2 - 3a^2)^3z^2 - 3a^2(z - 3a^2) - 1] = 0\]
\[(z^2 - 3a^2)^3z^2 - 3a^2(z - 3a^2) - 1 = 0\]  \hspace{1cm} (1)

Let \(w = z^2 - 3a^2\). We then get:

\[w^3(w + 3a^2) - 3a^2(w) - 1 = 0\]
\[w^4 - 3a^2w^3 - 3a^2w - 1 = 0\]  \hspace{1cm} (2)

For the moment, consider all two cycle points satisfying the equation, \(P_a(z) = -z\)
\[ z^3 - 3a^2 z = -z \]
\[ z^3 - 3a^2 z + z = 0 \]
\[ z(z^2 - 3a^2 + 1) = 0 \]
\[ z^2 - 3a^2 + 1 = 0 \]
\[ z^2 = 3a^2 - 1 \]
\[ z = \pm \sqrt{3a^2 - 1} \]

We now have five solutions: \( 0, \sqrt{3a^2 + 1}, -\sqrt{3a^2 + 1}, \sqrt{3a^2 - 1}, \) and \(-\sqrt{3a^2 - 1} \).

Notice, since \((z^2 - 3a^2 - 1)\) divides into (1), \(w - 1\) divides into (2).

Also, since \((z^2 - 3a^2 + 1)\) divides into (1), \(w + 1\) divides into (2).

These two facts imply that \((w - 1)(w + 1) = w^2 - 1\) will divide (2).

\[
\begin{array}{cccc}
  w^2 & -3a^2 w & +1 \\
\hline
  w^4 & -3a^2 & +0w^2 & -3a^2 w & -1 \\
  -w^4 & +0w^3 & +w^2 & & \\
 0 & -3a^2 w^2 & +w^2 & -3a^2 w & -1 \\
 +3a^2 w^2 & +0w^2 & +3a^2 w & & \\
 +w^2 & & & -1 \\
-w^2 & & & +1 \\
0 & & & & \\
\end{array}
\]
So, consider the equation:

\[ w^2 - 3a^2w + 1 = 0 \]

Using the quadratic formula, we then get,

\[ w = \frac{3a^2 \pm \sqrt{9a^4 - 4}}{2} \]

But, \( w = z^2 - 3a^2 \), so by substituting we get,

\[
\begin{align*}
  z^2 - 3a^2 &= \frac{3a^2 \pm \sqrt{9a^4 - 4}}{2} \\
  z^2 &= 3a^2 + \frac{3a^2 \pm \sqrt{9a^4 - 4}}{2} \\
  z^2 &= \frac{9a^2 \pm \sqrt{9a^4 - 4}}{2} \\
  z &= \pm \sqrt{\frac{9a^2 \pm \sqrt{9a^4 - 4}}{2}}
\end{align*}
\]

Therefore, we now have all nine points of period 2:

0, \( \sqrt{3a^2 + 1} \), \(-\sqrt{3a^2 + 1} \), \( \sqrt{3a^2 - 1} \), \(-\sqrt{3a^2 - 1} \), \( \sqrt{\frac{9a^2 + \sqrt{9a^4 - 4}}{2}} \), \( \sqrt{\frac{9a^2 - \sqrt{9a^4 - 4}}{2}} \), \(-\sqrt{\frac{9a^2 + \sqrt{9a^4 - 4}}{2}} \), \(-\sqrt{\frac{9a^2 - \sqrt{9a^4 - 4}}{2}} \), and \(-\sqrt{\frac{9a^2 - \sqrt{9a^4 - 4}}{2}} \)

Now, we must find when these periodic points are attracting. Notice that the first three cases are already established with the three fixed points. Before we move forward, however, we must describe how to determine which period-2 points are attracting.
Definition 3. A periodic point, \( p \), of period \( n \) is attracting when \( |(f^n)'(p)| < 1 \).

We also want to prove a small theorem that will make it easy for us to compute \( (f^n)'(p) \).

Theorem 2. If \( z_0 \) is a periodic point of period \( n \), then

\[
(f^n(z_0))' = f'(z_0) \cdot f'(z_1) \cdots f'(z_{n-1})
\]

where \( \{z_0, z_1, \ldots, z_{n-1}\} \) is the orbit of the periodic point \( z_0 \).

Proof. This proof is accomplished with repeated use of Chain Rule.

\[
(f^n(z_0))' = \left( f(\ldots f(z_0)) \right)'\underbrace{\ldots}_{n \text{ compositions}}
\]

\[
= \left( \underbrace{f'(f(\ldots f(z_0)))\ldots}_{n \text{ compositions}} \right) \cdot \left( f'(\ldots f(z_0)) \right)\underbrace{\ldots}_{n-1 \text{ compositions}}
\]

\[
= \left( \underbrace{f'(f(\ldots f(z_0)))\ldots}_{n \text{ compositions}} \right) \cdot \left( f'(\ldots f(z_0)) \right)\underbrace{\ldots}_{n-1 \text{ compositions}}
\]

\[
\cdots
\]

\[
= f'(f(\ldots f(z_0))) \cdots f'(f(z_0)) \cdots f'(z_0) \cdot f'(z_0)
\]

\[
(f^n(z_0))' = f'(z_{n-1}) \cdot f'(z_{n-2}) \cdots f'(z_1) \cdot f'(z_0)
\]
So, as long as we know the orbit of a periodic point, we can find where it is attracting. Notice that,

\[ P_a(\sqrt{3a^2 - 1}) = -\sqrt{3a^2 - 1} \]

Therefore, \( \sqrt{3a^2 - 1} \) is attracting when \( |f'(\sqrt{3a^2 - 1}) \cdot f'(-\sqrt{3a^2 - 1})| < 1 \).

\[
|f'(\sqrt{3a^2 - 1}) \cdot f'(-\sqrt{3a^2 - 1})| = |(3(\sqrt{3a^2 - 1})^2 - 3a^2) \cdot (3(-\sqrt{3a^2 - 1})^2 - 3a^2)|
\]
\[
= |(3(3a^2 - 1) - 3a^2)^2|
\]
\[
= |(9a^2 - 3 - 3a^2)^2|
\]
\[
= |(6a^2 - 3)|^2
\]
\[
|(6a^2 - 3)|^2 < 1
\]
\[
|(6a^2 - 3)| < 1
\]
\[
|a^2 - \frac{1}{2}| < \frac{1}{6}
\]

Let \( w = a^2 \). So:

\[
|w - \frac{1}{2}| < \frac{1}{6}
\]

This is a circle with center at \( \frac{1}{2} \) and radius \( \frac{1}{6} \). But we want \( a^2 \), not \( w \). So we have to take the square root of the circle \( |w - \frac{1}{2}| < \frac{1}{6} \). Doing so will yield two lobes “centered” at \( \frac{1}{\sqrt{3}} \) and \( -\frac{1}{\sqrt{3}} \), both tangent to the circle \( |a| < \frac{1}{\sqrt{3}} \) at the points \( \frac{1}{\sqrt{3}} \) and \( -\frac{1}{\sqrt{3}} \) respectively, and both with maximum “widths” at \( \frac{\sqrt{6}}{3} \) and \( -\frac{\sqrt{6}}{3} \) respectively.
Due to the symmetry that exists in this dynamical system, finding when 
\(-\sqrt{3a^2-1}\) is attracting will yield the same results.

There are now only four more critical points to consider. First, their orbit
must be determined. We can do this easily by substituting for a specific \(a\) value and then doing the algebra. Let \(a = 0\), we then get that

\[
\sqrt{\frac{9a^2 + \sqrt{9a^4 - 4}}{2}} = \sqrt{\frac{9(0)^2 + \sqrt{9(0)^4 - 4}}{2}} \tag{3}
\]
\[
= \sqrt{\frac{0 + \sqrt{0 - 4}}{2}}
\]
\[
= \sqrt{\frac{-4}{2}}
\]
\[
= \sqrt{\frac{2i}{2}} = i
\]

\[
\sqrt{\frac{9a^2 - \sqrt{9a^4 - 4}}{2}} = \sqrt{\frac{9(0)^2 - \sqrt{9(0)^4 - 4}}{2}} \tag{4}
\]
\[
= \sqrt{\frac{0 - \sqrt{0 - 4}}{2}}
\]
\[
= \sqrt{\frac{-\sqrt{-4}}{2}}
\]
\[
= \sqrt{\frac{-2i}{2}} = \sqrt{-i} = i\sqrt{i}
\]

\[
-\sqrt{\frac{9a^2 + \sqrt{9a^4 - 4}}{2}} = -\sqrt{\frac{9(0)^2 + \sqrt{9(0)^4 - 4}}{2}} \tag{5}
\]
\[
= -\sqrt{\frac{0 + \sqrt{0 - 4}}{2}}
\]

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\[-\sqrt{9a^2 - \sqrt{9a^4 - 4}} = -\sqrt{9(0)^2 - \sqrt{9(0)^4 - 4}} \]

\[-\frac{9a^2 - \sqrt{9a^4 - 4}}{2} = -\sqrt{\frac{0 - \sqrt{0 - 4}}{2}} = -\sqrt{-\frac{-2i}{2}} = -\sqrt{-i} = -i\sqrt{i} \tag{6}\]

Now, by choosing one value and running through the function I will reveal the orbits of the remaining critical points.

Let \( z = \sqrt{i} \),

\[ P_0(\sqrt{i}) = (\sqrt{i})^3 - 3(0)^2(\sqrt{i}) \]

\[ = (\sqrt{i})^3 \]

\[ = i\sqrt{i} = (4) \]

So, (3) maps to (4). Leaving (5) to map to (6), but we will verify this quickly

\[ P_0(-\sqrt{i}) = (-\sqrt{i})^3 - 3(0)^2(-\sqrt{i}) \]

\[ = (-\sqrt{i})^3 \]

\[ = -i\sqrt{i} = (6) \]
Now, all that is left to find is when they’re attracting. We’ll begin with (3) and (4).

\[ |f'(\sqrt{\frac{9a^2 + \sqrt{9a^4 - 4}}{2}}) \cdot f'(\sqrt{\frac{9a^2 - \sqrt{9a^4 - 4}}{2}})| < 1 \]

\[ |[3(\sqrt{\frac{9a^2 + \sqrt{9a^4 - 4}}{2}})^2 - 3a^2] \cdot [3(\sqrt{\frac{9a^2 - \sqrt{9a^4 - 4}}{2}})^2 - 3a^2]| < 1 \]

\[ |[3(\frac{9a^2 + \sqrt{9a^4 - 4}}{2}) - 3a^2] \cdot [3(\frac{9a^2 - \sqrt{9a^4 - 4}}{2}) - 3a^2]| < 1 \]

\[ |[\frac{21a^2 + 3\sqrt{9a^4 - 4}}{2}] \cdot [\frac{21a^2 - 3\sqrt{9a^4 - 4}}{2}]| < 1 \]

\[ |\frac{360a^4 + 36}{4}| < 1 \]

\[ |9a^4 + 9| < 1 \]

\[ |a^4 + 1| < \frac{1}{9} \]

Let \( w = a^4 \). So:

\[ |w + 1| < \frac{1}{9} \]

This is a circle with center at \(-1\) and radius \( \frac{1}{9} \). But we want \( a^4 \), not \( w \). So we have to take the fourth root of the circle \(|w + 1| < \frac{1}{9}\). Doing so will yield four lobes tangent to the four second largest lobes (see figure 8). Now we’ll
finish with (5) and (6).

\[
|f(-\sqrt{\frac{9a^2 + \sqrt{9a^4 - 4}}{2}}) \cdot f(-\sqrt{\frac{9a^2 - \sqrt{9a^4 - 4}}{2}})| < 1
\]

\[
|[3(-\sqrt{\frac{9a^2 + \sqrt{9a^4 - 4}}{2}})^2 - 3a^2] \cdot [3(-\sqrt{\frac{9a^2 - \sqrt{9a^4 - 4}}{2}})^2 - 3a^2]| < 1
\]

\[
|[3(\frac{9a^2 + \sqrt{9a^4 - 4}}{2}) - 3a^2] \cdot [3(\frac{9a^2 - \sqrt{9a^4 - 4}}{2}) - 3a^2]| < 1
\]

\[
|\frac{21a^2 + 3\sqrt{9a^4 - 4}}{2} \cdot \frac{21a^2 - 3\sqrt{9a^4 - 4}}{2}| < 1
\]

\[
|\frac{360a^4 + 36}{4}| < 1
\]

\[
|9a^4 + 9| < 1
\]

\[
|a^4 + 1| < \frac{1}{9}
\]

So we have a double cover of the lobes that we had just found in the previous equations. This is expected due to having already found the final four period-2 lobes after evaluating where (3) and (4) were attracting.

These are all of the lobes of the connectedness locus that I have completely categorized algebraically. In the appendix of figures I’ve included the locations of some period-3 lobes (see figure 9) that I’ve found only numerically. I also have included a few Julia sets along with some descriptions of their dynamics (see figures 3 and 4). Although not all images are referenced in the paper, I do encourage the interested reader to look them all over.

With this, my study of abstract dynamical systems, focusing on complex
cubic polynomials, comes to a temporary conclusion. I’ve enjoyed the material that I’ve learned and the work that I’ve done so thoroughly that there is no doubt in my mind that I will continue to educate myself in this field of Mathematics.

“It can be argued that the mathematics behind these images [of the orbit diagram for quadratic functions and the Mandelbrot set] is even prettier than the pictures themselves.”

References


